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Force transmissibility of structures traversed by a moving system

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Abstract

This paper addresses a general method of computing the force transmissibility for complex or statically indeterminate mechanical system with more than one support. Moreover, the forces transmitted to the ground when one mechanical system travels across another are calculated. The results of an oscillator travelling over a beam on three simple supports is addressed in order to illustrate the method.

1. Introduction

Systems that interact through time-varying interfaces can be found in different situations such as, vehicles moving over bridges or other flexible foundations, cutters moving in machining problems, car tyres travelling over rough surfaces, etc. Such class of problem typically include lumped or distributed parameter systems that interact through moving contacts. Taking vehicles travelling over a bridge as an example, the response depends on the dynamic properties of the traversing vehicles and of the bridge, the vehicle speed and the surface roughness. In order to control the bridge structure vibration it is important to be able to accurately predict the bridge response to the moving vehicle and the resulting responses of the vehicle [1]. In particular, it is important to predict the transmitted force if the aim of the applied control is to reduce the transmitted force to the ground. To the knowledge of the authors, there are no studies that aim to find the force transmitted through bridge supports, probably because of the limited literature available on structural force transmissibility.

The problem of moving vehicles interacting with highways infrastructures has been examined by many researchers, mainly driven by concerns over fatigue [2]. In general, three types of problems are reported in the literature: there are moving force [3], moving mass [4] and moving oscillator [5] problems. A vehicle can be modelled as a moving force when its inertia is negligible, compared to that of the crossed system, since no dynamic interaction between the two systems is taken into account. When the inertia cannot be neglected a moving mass model should be employed. The inclusion of both inertia and interaction forces in the moving oscillator problem can be found only in recent studies on vehicle–bridge interactions [1].

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The problem of the vibration of solids and structures subjected to moving loads has been broadly discussed by Frýba in his classical book [5]. The unusual feature of moving load is that they are variable in both time and space, which make these problems quite difficult to approach. For simple structures with simple boundary conditions and crossed by moving load the method of expansion in normal modes is suggested. Fourier transformations are used for the length coordinates and Laplace–Carson transformations for time coordinate. More complicated structures were instead approached by the generalised method of finite integral transformations, which leads to results formally identical with the method of expansion in normal modes. Such methods have the advantage of a comparatively easy solution and the shortcoming of having a slow convergence in some instances, especially at high speed. Infinite structures are solved considering only the steady-state vibration of the body as the load moves from minus infinity to plus infinity. This method is advantageous in that it supplies solution in closed form; however, the solution in many cases is difficult to attain.

In the past two decades, researcher have developed vehicle models of various complexity to account for the dynamic properties of the vehicle and accurately predict the amplification of deflections and stresses in the bridge structure, and more recently a FEM dynamic analysis has been developed by Yang et al. [6].

Tsao et al. [7] addressed a control-oriented formulation of structures interacting through moving contact points. The example they used for illustration is the dynamic interaction of moving vehicles over bridge structures. Such systems are time varying due to the moving vehicles. The system representation has been partitioned into a time invariant part (bridge and vehicle dynamics) and a time-varying part (moving contact points). Such a systemic approach facilitates a modularised analytical model development, analysis, simulation, and control system design. Further, such problems may be put in a feedback form in order to exploit classical numerical and control tools. Due to its modularity such approach seems the most suitable for solving the force transmissibility problems as well. The force transmitted to the ground, due to the interaction with a moving structure, can most easily be obtained from force transmissibility matrices, which are much less widely studied than velocity transmissibility matrices.

The velocity transmissibility is the ratio between the resulting velocity V_A of the structure at the point A and the velocity imposed at the support B. It is defined in many general books on structural dynamics [8]. Ribeiro et al. [9] generalised the transmissibility concept to structures with several degrees of freedom. They showed that a velocity transmissibility matrix can be calculated in terms of the mobility matrices of the structure. In particular, the velocity transmissibility between two sets of point A and B can be obtained by experimentally exciting the structure at at least as many point as in A, if the mobility matrices between excitation and sets A and B are known. A surprising consequence of this is that each transmissibility matrix is then not only characteristic of the structure but also of the way the structure is loaded.

The force transmissibility T_F, as defined by Ungar [10] is the ratio between the resulting force F_B acting on a support B of the structure and the force F_A acting on a point A of the same structure. As a direct consequence of the superposition and reciprocity principles, velocity and force transmissibility are shown to be equal when force and velocities act in the same direction on a linear, time-invariant single-degree-of-freedom (1dof) system [10].

The aim of this paper is twofold. Firstly, the authors want to generalise the concept to force transmissibility to multi degree of freedom structures and show the relationship between force and velocity transmissibilities. Secondly, we propose to use the convolution theorem to predict the force transmitted to the ground from structures crossed by moving flexible body. The method proposed by Tsao et al. [7] has been then slightly modified. A numerical example is then presented, in order to illustrate the problem in more detail.

2. Transmissibility frequency functions

In order to define velocity and force transmissibility a general elastic body may be considered. As shown in Fig. 1, input and output velocities and forces are applied to the body. Two sets of points are chosen on the body, labelled with subscript A and B. The force transmissibility matrix $\mathbf{T}_F$ between the output force vector, $-\mathbf{f}'_B$, and the input force vector, $\mathbf{f}_A$, is defined as the matrix that satisfies the following relation:

$$-\mathbf{f}'_B = \mathbf{T}_F \mathbf{f}_A \tag{1}$$

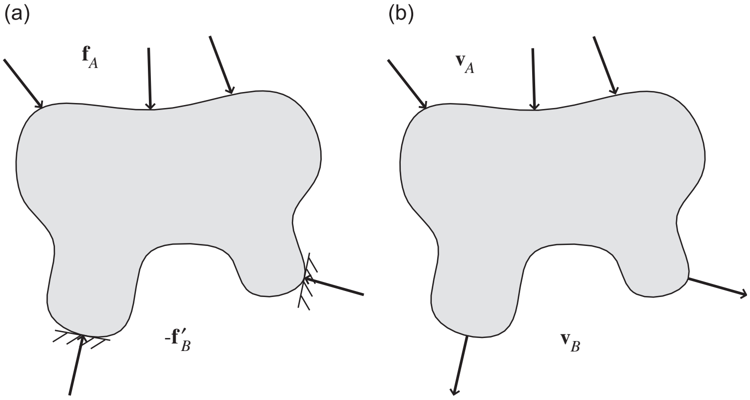

Fig. 1. Forces applied to a constrained body for force transmissibility definition (a) and velocities of a free body for velocity transmissibility (b).

where the apostrophe on the output force is to indicate that such a force vector can be obtained only by putting constraints at this points, i.e., $-\mathbf{f}'_B$ coincides with the force transmitted from the body to the supports. The transmissibility is defined in the frequency domain, but the explicit dependence on the frequency ω will be dropped for notational convenience.

The force transmissibility can be measured by placing force transducers directly to the blocked structure or can be obtained numerically from the mobilities of the free-standing body. Taking into account the body in free-standing condition, the output velocity vector $\mathbf{v}_B$ is given by

$$\mathbf{v}_B = \mathbf{Y}_{BA}\mathbf{f}_A + \mathbf{Y}_{BB}\mathbf{f}_B, \tag{2}$$

where $\mathbf{Y}_{BA}$ is the cross-mobility matrix between input and output points of the free standing body and $\mathbf{Y}_{BB}$ is the point mobility matrix at the output points of the free standing body. The input velocity vector $\mathbf{v}_A$ can be similarly expressed as

$$\mathbf{v}_A = \mathbf{Y}_{AA}\mathbf{f}_A + \mathbf{Y}_{AB}\mathbf{f}_B, \tag{3}$$

where $\mathbf{Y}_{AB}$ is the cross-mobility matrix between output and input points of the free standing body and $\mathbf{Y}_{AA}$ is the point mobility matrix at the input points of the free-standing body.

If constraints are applied to the body, as shown in Fig. 1(a), i.e., $\mathbf{v}_B = 0$, the force transmissibility can be obtained from Eq. (2) as

$$\mathbf{T}_F = \mathbf{Y}_{BB}^{-1}\mathbf{Y}_{BA}, \tag{4}$$

where the mobility matrix $\mathbf{Y}_{BB}$ has been assumed to be nonsingular. Since Eq. (4) is obtained from free-standing mobilities, it enables us to deal easily with statically indeterminate structures. The force transmitted to the supports, due to nonstationary forces, can then be found using the convolution theorem, as explained in Section 3.

Force transmissibilities are always related to constrained structures and for very large structures it is not possible to measure the mobility on the free-standing body and only in-situ measurements are possible. It will be shown that is not possible to obtain the matrix $\mathbf{T}_F$ from only in-situ measurements. When supports are placed at the output points, the point mobility matrix of the constrained structure, $\mathbf{Y}'_{AA}$, may be obtained from Eqs. (1), (3) and (4), i.e.,

$$\mathbf{Y}'_{AA} = \mathbf{Y}_{AA} - \mathbf{Y}_{AB}\mathbf{Y}_{BB}^{-1}\mathbf{Y}_{BA} = \mathbf{Y}_{AA} - \mathbf{Y}_{AB}\mathbf{T}_F. \tag{5}$$

The force transmissibility matrix can then be expressed as a function of the point mobility matrix of the constrained structure, provided that $\mathbf{Y}_{AB}$ is square and nonsingular:

$$\mathbf{T}_F = -\mathbf{Y}_{AB}^{-1}(\mathbf{Y}'_{AA} - \mathbf{Y}_{AA}). \tag{6}$$

From Eq. (6) it can be inferred that the force transmissibility matrix can be obtained from in-situ measurements (on the supported structure) only if the mobility matrices $\mathbf{Y}_{AB}$ and $\mathbf{Y}_{AA}$ of the free structure are known already, that is from a numerical model or from experimental measurements.

The relationship between force and velocity transmissibility is known for 1dof systems [10]. Although it is not directly related to the main topic of this paper, we briefly discuss the relationship between $\mathbf{T}_F$ and $\mathbf{T}_V$. The velocity transmissibility matrix $\mathbf{T}_V$ is defined by the following relationship:

$$\mathbf{v}_A = \mathbf{T}_V \mathbf{v}_B, \tag{7}$$

where $\mathbf{v}_A$ is the input velocity vector and $\mathbf{v}_B$ is the output velocity vector. By definition, the velocity transmissibility is obtained by freeing the B coordinates, at which is applied the force $\mathbf{f}_B$, and measuring the velocity at points B and A, where no force is applied, hence $\mathbf{f}_A = \mathbf{0}$. Using this condition and substituting Eqs. (2) and (3) in Eq. (7) a new expression of the velocity transmissibility matrix can be found as

$$\mathbf{T}_V = \mathbf{Y}_{AB}\mathbf{Y}_{BB}^{-1} \tag{8}$$

and using Eq. (4) the relationship between force and velocity transmissibilities can be found as

$$\mathbf{T}_V \mathbf{Y}_{BA} = \mathbf{Y}_{AB}\mathbf{T}_F. \tag{9}$$

Using the reciprocity principle, i.e., $\mathbf{Y}_{AB} = \mathbf{Y}_{BA}^{\mathrm{T}}$, this becomes

$$\mathbf{T}_V \mathbf{Y}_{AB}^{\mathrm{T}} = \mathbf{Y}_{AB}\mathbf{T}_F, \tag{10}$$

which shows that the force transmissibility between points A and B is equal to the velocity transmissibility between points B and A, only in a 1dof system, as shown by Ungar [10]. In the case where the sets of input and output point A and B coincide, the matrices $\mathbf{T}_F$, $\mathbf{T}_V$ and $\mathbf{Y}_{AB}$ are square and an equivalence relation of similarity between the force and velocity transmissibility can be found [11]. Such equivalence implies that the similar matrices represent the same linear transformation.

3. Description of the numerical method

In this section, the problem of a base structure with a moving flexible system is considered in order to find the forces transmitted to the ground through the supports. The full system is made up of two sub-systems: a grounded base system, considered as a distributed parameters system modelled by an elemental formulation; and a moving system, considered for simplicity to be composed by N_o lumped parameter systems that may be called oscillators. The two systems are connected to each other at one point for each travelling system. The two systems can be represented using a feedback control block diagram, and their interaction through moving contact points can then be represented as shown in Fig. 2.

The force transmissibility functions can be used to obtain the forces on the supports, $-\mathbf{f}'_B(t)$, as

$$-\mathbf{f}'_B(t) = \mathfrak{F}^{-1}[\mathbf{T}_F(\omega)] \otimes \mathbf{f}_A(t), \tag{11}$$

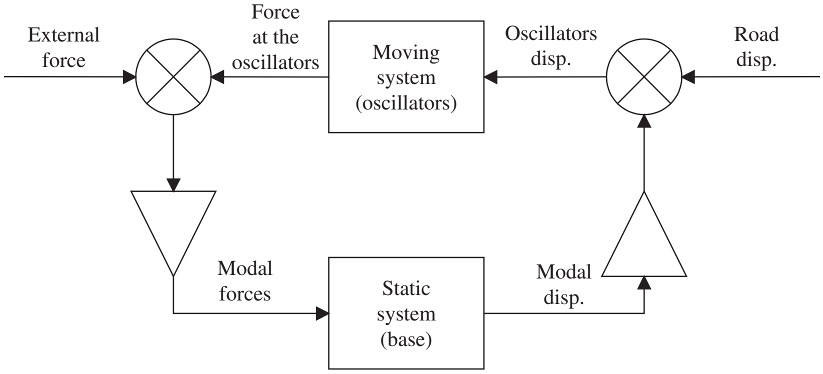

Fig. 2. Feedback diagram of static and moving systems.

where $\mathfrak{F}^{-1}$ is the inverse Fourier transform and $\mathbf{f}_A(t)$ is the force acting on the base structure due to the moving oscillator. The main problem is then reduced to finding the dynamic force vector $\mathbf{f}_A(t)$ due to the interaction between base and moving structure. This problem can be addressed by analysing two structures interacting through moving contact points.

Such an analysis is similar to that proposed by Tsao et al. [7]. Both sub-systems need to be modelled and a state-space representation has been chosen for this here. The inputs of the base system are the forces applied to it and the output are the displacement at the nodal points. The travelling load has as input the displacement of its base and as output the force generated at his bottom. The interaction between two systems may then be represented as a feedback loop, shown in Fig. 2, where the interacting force and motion at the contact points are the input and output variables of the two sub-systems. Such an approach facilitates the development, analysis, simulation and control design of such time-varying system. Moreover, the displacement at the base of the travelling system can be updated with the road displacement at that point, before it is sent to the moving sub-system as an input. At the same way the forces at the oscillators can be updated with some external forces applied to them, as for example the oscillator weights, before they are sent to the base system as input. Since the contact points are moving some matrix multiplication, explained in detail in Section 4.1, is needed.

The state-space equation of the base system are usually expressed in generalised coordinates. Therefore, the forces acting on it should be multiplied by the mode shape matrix. The generalised force $p_n(t)$ applied to the nth mode is given by

$$p_n(t) = \sum_{q=1}^{N_m} \psi_n(\zeta_q(t))F_q(t), \tag{12}$$

where t is the time variable, ψ_n is the nth mode shape and $\zeta_q(t)$ is the coordinate of the qth point on the structure, that is divided in N_m points. A vector representation of Eqs. (12) is given by

$$\mathbf{p}(t) = \mathbf{\Psi}^{\mathrm{T}}\mathbf{F}(t), \tag{13}$$

where $\mathbf{\Psi}$ is the mode shape matrix defined in the N_m points of the base structure.

The base system can be represented with the state-space equation

$$\dot{\mathbf{q}}(t) = \mathbf{\Omega q}(t) + \mathbf{Bp}(t), \tag{14}$$

where the state vector $\mathbf{q}$ is a vector of modal displacement and velocities. The real displacement $\mathbf{w}$ is a vector given by

$$\mathbf{w}(t) = \mathbf{\Psi}(\boldsymbol{\zeta}(t))\mathbf{q}(t). \tag{15}$$

Note that the system is time invariant but since the position of the point forces is changing with time, the generalised forces are changing with time. In other words, each system is linear but the moving contact makes the global system nonlinear.

The ith vehicle dynamics may be represented by a state-space model with a displacement input vector $\mathbf{y}_i$ and the force output vector $\mathbf{f}_i$:

$$\dot{\mathbf{x}}_i = \mathbf{A}_i\mathbf{x}_i + \mathbf{B}_i\mathbf{y}_i, \tag{16}$$

$$\mathbf{f}_i = \mathbf{C}_i\mathbf{x}_i + \mathbf{D}_i\mathbf{y}_i, \tag{17}$$

where $\mathbf{x}_i$ is the state vector.

If N_o travelling systems are crossing the steady system, this is excited by k point forces, due to the ith moving oscillator located at position $\zeta_i(t)$, along the reference axis. The parameters of the travelling system are also time invariant but the location at which they are connected with the elastic body is varying with time. The input of one oscillator y_i is the displacement of the base body at the contact point ζ_i plus an external disturbance $r(t)$, which represents the road surface profile

$$y_i(t) = w(\zeta_i(t), t) + r(\zeta_i), \tag{18}$$

where $w(\zeta_i, t)$ can be obtained from Eqs. (15). Eq. (18) supposes the oscillator and the road surface to be in contact at all times.

The input of the steady body with respect to the ith oscillator is the output dynamic force of the oscillator system $f_i(t)$ plus any external disturbance $d_i(t)$:

$$F_i(t) = f_i(t) + d_i(t). \tag{19}$$

Since the force $f_i(t)$ usually does not take into account the vehicle static gravity, this can be taken into account in the disturbance term considering the relative force at the base of the oscillator: $d_i(t) = M_i g$.

4. Application to a 1dof oscillator crossing a 3-hinged beam

4.1. Mathematical model

In order to show the potential of the proposed methodology a practical application is now taken into account. In particular, the problem of an oscillator travelling over a three-hinged beam, shown in Fig. 3, is described. The three-hinged beam is a very simple statically indeterminate structure and illustrates the advantages of this approach. Such a problem can be seen as a very simple model of a train wheel passing over a railway track.

The two sub-systems, the 1dof oscillator and the three-hinged beam, have been chosen to be modelled in the state-space. First the oscillator has been modelled as shown in Fig. 4(a), with some viscous damping. The oscillator can be considered as a 1dof system with imposed displacement at the base. Defining y as the displacement imposed to the base with respect to the ground reference system, and x as the displacement of the oscillator mass with respect to the same reference system, the relative displacement z between the base and the mass may be defined as $z = x - y$, as shown in Fig. 4(a). The system dynamics may be expressed by

$$m\ddot{x} + c\dot{z} + kz = 0, \tag{20}$$

where m is the mass of the oscillator and c and k are the viscous damping and the stiffness coefficient of the oscillator. Choosing the displacement and the velocity of the oscillator mass as state variables, its state-space description is given by

$$\begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & 0 \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ \dfrac{k}{m} & \dfrac{c}{m} \end{bmatrix} \begin{Bmatrix} y \\ \dot{y} \end{Bmatrix}, \tag{21}$$

where the input of the state-space model are the displacement and velocity of the oscillator base. The output of the state-space-system must be the force f applied from the oscillator to the base support, i.e.,

$$f = [k \;\; c] \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} + [-k \;\; -c] \begin{Bmatrix} y \\ \dot{y} \end{Bmatrix}. \tag{22}$$

The beam has been modelled as an elemental system formed by N_m points between which the two structure interacts, as shown in Fig. 4(b). It has been described in the state-space domain using the modal decomposition. As already seen the dynamics of the beam can be expressed as the superposition of N_m 1dof systems using the modal transformation [12]

$$\mathbf{w} = \mathbf{\Psi q},$$

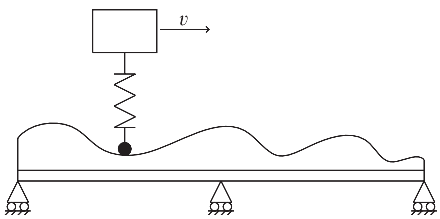

Fig. 3. A 1dof oscillator moving across a three-hinged beam.

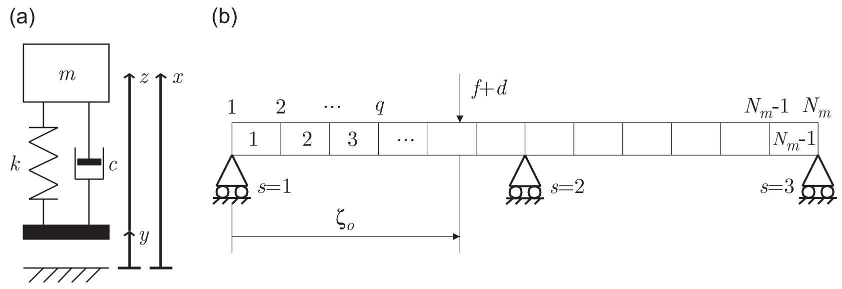

Fig. 4. The 1dof oscillator (a) and the beam elemental model (b).

where $\mathbf{w}$ is the real displacement vector, $\mathbf{q}$ is the modal vector and $\mathbf{\Psi}$ is the modal matrix. The dynamics of the system may be then expressed in matricial form as

$$\mathbf{\Psi}^{\mathrm{T}}\mathbf{M}\mathbf{\Psi}\ddot{\mathbf{q}} + \mathbf{\Psi}^{\mathrm{T}}\mathbf{K}\mathbf{\Psi}\mathbf{q} = \mathbf{\Psi}^{\mathrm{T}}\mathbf{f}, \tag{23}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{k}$ is the stiffness matrix and $\mathbf{f}$ is vector of the applied forces.

If the mode shapes are mass normalised (that is $\mathbf{\Psi}^{\mathrm{T}}\mathbf{M}\mathbf{\Psi} = \mathbf{I}$ and $\mathbf{\Psi}^{\mathrm{T}}\mathbf{K}\mathbf{\Psi} = \mathrm{diag}(\omega_n^2)$), N_m second-order uncoupled differential equations can be obtained, in the form of

$$\ddot{q}_n + \omega_i^2 q_n = p_n \quad \text{for } n = 1, 2, \ldots, N_m, \tag{24}$$

where ω_n is the natural frequency of the nth mode and p_n are the modal forces obtained as elements of the modal force vector $\mathbf{p} = \mathbf{\Psi}^{\mathrm{T}}\mathbf{f}$.

Some damping was then added to the structure in the form of Rayleigh damping, i.e., the damping matrix is proportional to the mass and stiffness matrices. Such a damping has been used in order to maintain N_m uncoupled equations, that now are expressed by

$$\ddot{q}_n + 2\xi_n\omega_n\dot{q}_n + \omega_n^2 q_n = p_n \quad \text{for } n = 1, 2, \ldots, N_m, \tag{25}$$

where ξ_n is the nth mode damping ratio, related to the nth mode damping coefficient c_n by

$$\xi_n = \frac{c_n}{2\omega_n}. \tag{26}$$

Eqs. (25) can be expressed in a matrix form as

$$\ddot{\mathbf{q}} + 2\mathbf{\Xi}\mathbf{\Omega}\dot{\mathbf{q}} + \mathbf{\Omega}^2\mathbf{q} = \mathbf{p}, \tag{27}$$

where $\mathbf{\Xi} = \mathrm{diag}(\xi_n)$ and $\mathbf{\Omega} = \mathrm{diag}(\omega_n)$.

If follows that the system can be expressed in the state-space form as

$$\begin{Bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{Bmatrix} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{\Omega}^2 & -2\mathbf{\Xi}\mathbf{\Omega} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{Bmatrix} + \begin{bmatrix} 0 \\ \mathbf{I} \end{bmatrix} \mathbf{p}, \tag{28}$$

where the input of the system are the modal forces $\mathbf{p}$ and the state variables are the modal beam displacement $\mathbf{q}$ and velocity $\dot{\mathbf{q}}$.

At this point the two systems can be connected as shown in Fig. 2. The new input of the beam system is given from the combination of the external force d and the dynamic force f, applied at the oscillator base:

$$\mathbf{p} = \mathbf{\Psi}^{\mathrm{T}}[\boldsymbol{\tau}_{\mathrm{pf}}(\zeta_o)(f + d)], \tag{29}$$

where $\mathbf{\Psi}$ is the beam mode shape matrix at fixed locations and $\boldsymbol{\tau}_{\mathrm{pf}}(\zeta_o)$ is a participation factor vector that takes into account the fact that the forces d and f are applied in a particular point of the beam, with coordinate ζ_o, which depends on the position of the oscillator at that time. Since the model is elemental, and not continuous, the standard modal projection cannot be used. The participation factor vector has been built so to linearly distribute the oscillator force between the two ends of the element over which the force is applied.

The input of the oscillator system is given by the sum of the beam displacement at the actual position of the oscillator ζ_o and the road profile at that point $r(\zeta_o)$:

$$y = \boldsymbol{\tau}_{\mathrm{pf}}(\zeta_o)^{\mathrm{T}}\boldsymbol{\Psi}\mathbf{q} + r(\zeta_o), \tag{30}$$

while the velocity of the oscillator base is given by the beam velocity and the transport velocity due to the road profile, i.e.,

$$\dot{y} = \boldsymbol{\tau}_{\mathrm{pf}}^{\mathrm{T}}\boldsymbol{\Psi}\dot{\mathbf{q}} + \left(\frac{\mathrm{d}\boldsymbol{\tau}_{\mathrm{pf}}^{\mathrm{T}}}{\mathrm{d}\zeta_o}\boldsymbol{\Psi}\mathbf{q} + \frac{\mathrm{d}r}{\mathrm{d}\zeta_o}\right)v, \tag{31}$$

where v is the translational velocity of the oscillator over the beam. In the previous equations the participation vector $\boldsymbol{\tau}_{\mathrm{pf}}(\zeta_o)$ has been used to obtain the displacement ζ_o by linear interpolation of the displacements of the two points of the beam element over which the oscillator is travelling. Another advantage of the method is that any road profile can be considered.

The full system can be solved numerically, as explained in the next subsection, and the force vector $\mathbf{f}_A(t)$ acting over the three hinged beam can be found. The force at the sth support, $f_{As}(t)$ can be found applying the convolution theorem as

$$f_{As}(t) = \sum_{q=1}^{N_m} \mathfrak{F}^{-1}[T_{F,sq}(\omega)] \otimes f_q(t) \quad \text{for } s = 1, 2, 3, \tag{32}$$

where $\mathfrak{F}^{-1}[\cdot]$ is the inverse Fourier transformation, $f_q(t)$ is the force applied at the qth mode of the beam and $T_{F,sk}(\omega)$ is the force transmissibility between the input qth node of the base structure and the support s that can be obtained from Eq. (4) as an element of the force transmissibility matrix $\mathbf{T}_F$.

4.2. Numerical example

The state-space systems described previously have been solved numerically by means of SIMULINK®. The dimensions assumed for the beam and the oscillator are shown in Table 1. Initially, two velocities of the oscillator have been taken into account, 1 and 10 $\mathrm{m\,s^{-1}}$, that is the oscillator takes respectively 500 ms and 50 ms to traverse the beam length. The natural frequency of the oscillator is $f_o \approx 50$ Hz. The first anti-symmetrical natural frequency of the beam is $f_{a1} \approx 376$ Hz while the first symmetrical resonance is $f_{s1} \approx 588$ Hz. The critical speed is defined as the lowest speed at which the displacement of a beam subjected to a constant moving load continually increases during the travelling time. The critical speed $v_{\mathrm{cr.}}$ of a simple

Table 1
Properties of the system used for the simulation

Description and dimensions	Constant	Value
Length of the beam (m)	L	0.5
Beam broadness (m)	B	0.05
Beam thickness (m)	h	0.01
Young's modulus (Pa)	E	210×10^9
Beam density ($\mathrm{kg\,m^{-3}}$)	ρ	7800
Beam damping ratio ()	ξ	0.02
Oscillator mass (kg)	m	1
Oscillator spring stiffness ($\mathrm{Nm^{-1}}$)	k	0.1×10^6
Oscillator damping ratio ()	$\xi = \dfrac{c}{2\sqrt{km}}$	0.1
Oscillator velocity ($\mathrm{m\,s^{-1}}$)	v	1, 10, $v_{\mathrm{cr.}}$
Bump height (m)	b	0.5×10^{-3}

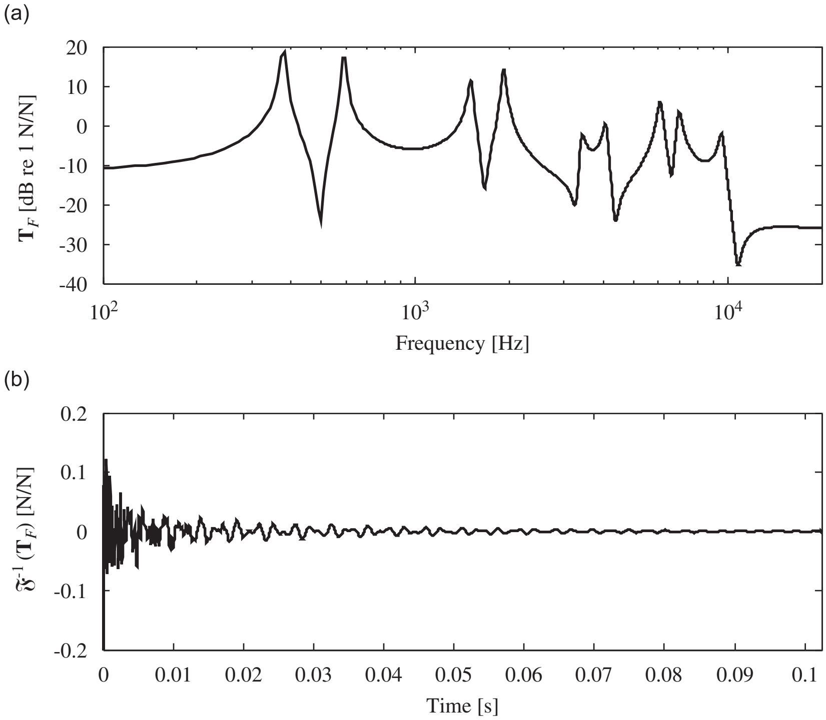

Fig. 5. Force transmissibility (a) and its inverse Fourier transform (b) between an excitation applied between the first two supports and the first support.

supported beam under a moving load, is [5]

$$v_{\mathrm{cr.}} = \frac{\pi}{L}\sqrt{\frac{EI}{\rho A}}. \tag{33}$$

In order to have and idea of the critical velocity of a three supports beam the gap between the first two supports may be considered, so to obtain $v_{\mathrm{cr.}} \approx 188\,\mathrm{m\,s^{-1}}$, which, as expected, is equal to $f_{a1}L/2$.

In order to find the force transmitted to the support by the travelling oscillator is necessary firstly to find the transmissibility matrix, given by Eq. (4), and then the inverse Fourier transformation of its elements, as can be seen in Eq. (32). Fig. 5(a) shows the transmissibility of the beam when a static harmonic excitation force is applied at the mid point between the first and second supports and when the output force is gathered at the first support. Fig. 5(b) shows the Green function of the force at the first support when a unity impulse is applied at the same point.

Fig. 6(a) shows the calculated forces transmitted to the three supports when the oscillator moves across the beam, with the dimensions and properties listed in Table 1, at a velocity $v = 1\,\mathrm{m\,s^{-1}}$. Since the velocity of the oscillator is very slow, that is $v \ll v_{\mathrm{cr.}}$, the problem is comparable to a quasi-static problem and the dynamic of the beam and oscillator can be neglected. As the oscillator arrives on the first support all its weight (10 N) is borne by this. The force on the first support then decreases to be near zero as the weight reaches the third support. The force on the first support is negative as the oscillator moves between the second and third support since the deformation of the beam, imposed by the weight of the oscillator, tends to lift the beam from the first support. The force over the second support starts from zero when the oscillator is over the first support, reaches the maximum, 10 N, at the second support, and becomes zero at the third support. In fact when the oscillator is over a support all the weight of the oscillator is borne from that support while the other two supports have to bear only the dynamic forces due to the beam oscillations if there are any.

In order to show the effect of the road profile a bump has been added to the model. Fig. 6(b) shows the results when the same oscillator travels over the beam with a bump of 0.5 mm placed in the middle, between

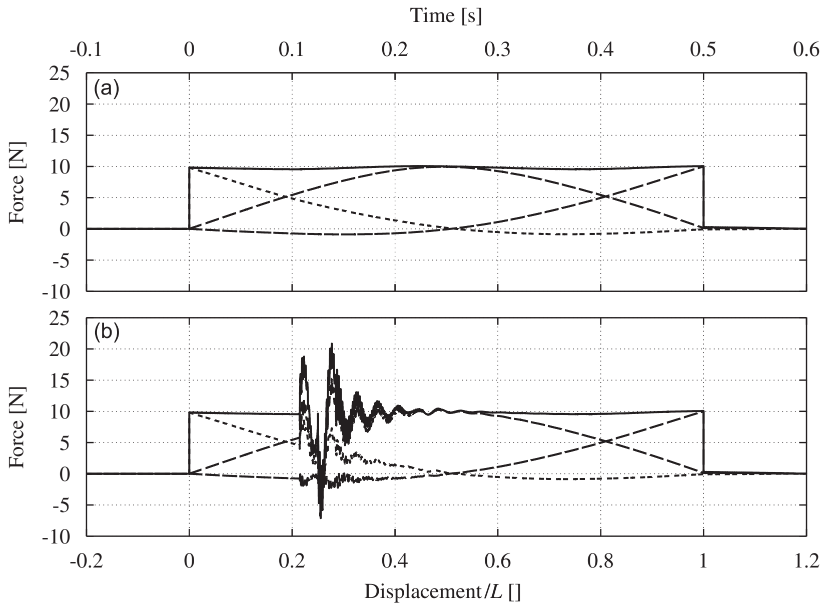

Fig. 6. Numerical force at the three supports when the oscillator travels with a velocity $v = 1\,\mathrm{ms}^{-1}$ over a plane beam (a) or over a beam with a small bump (b): ··· first support, — · — · — second support, - - - third support and — sum of the three forces.

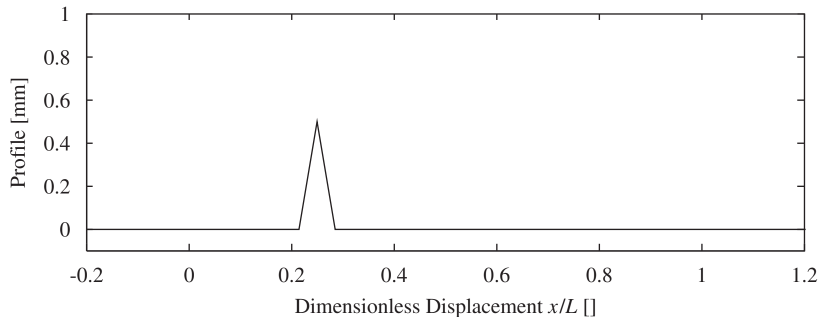

Fig. 7. The bump profile.

the first and second supports. The profile of the bump is shown in Fig. 7. When the oscillator hits the bump the forces at the first and second supports are increased. After the bump, and before reaching the second support, the forces at all three supports are affected by the oscillating dynamic forces generated by the beam vibrations. The largest oscillations have a frequency of 50 Hz, and are due to the the oscillator. A smaller oscillation, at about 560 Hz, can be compared to the first symmetrical beam mode $f_{s1} \approx 588\,\mathrm{Hz}$. After the oscillator has passed the second support the oscillation are damped out by the internal damping of the beam.

Fig. 8(a) shows the forces transmitted to the three supports when the oscillator moves across the beam at a greater velocity, $v = 10\,\mathrm{ms}^{-1}$. As the oscillator arrives on the first support its weight excites the beam vibrations and the forces oscillate at a frequency of about 625 Hz. Beside this oscillation, the force shapes are similar to those obtained with the slower velocity. Fig. 8(b) shows the results when the bump is added to the beam profile. When the oscillator hits the bump the forces at the first and second supports are increased significantly. The force at the first support jumps to about 40 N while that one at the second support jumps to about 45 N. The force at the third support is negative and reaches a magnitude of $-25\,\mathrm{N}$. A big increase on each force can thus be observed when the oscillator travels across the bump at this speed.

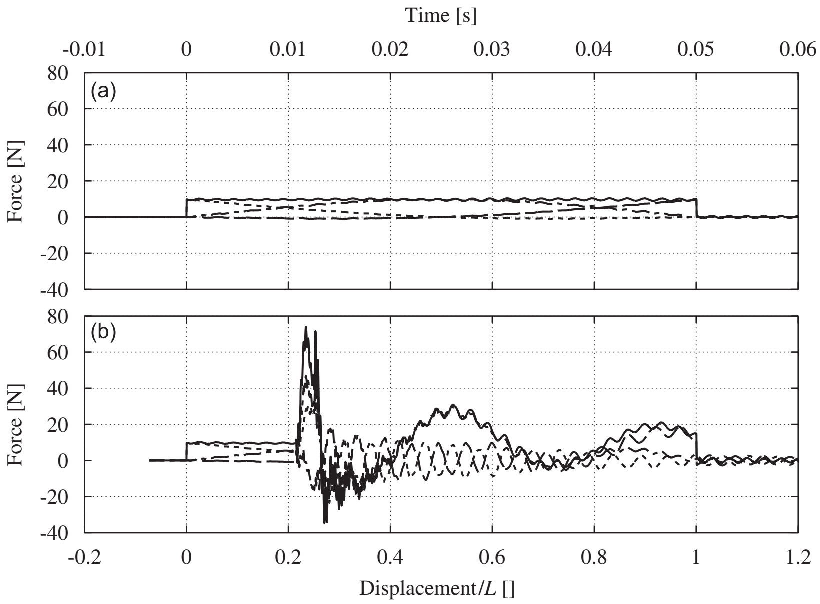

Fig. 8. Numerical force at the three supports when the oscillator travels with a velocity $v = 10\,\mathrm{ms}^{-1}$ over a plane beam (a) or over a beam with a small bump (b): ··· first support, – · – · – second support, - - - third support and — sum of the three forces.

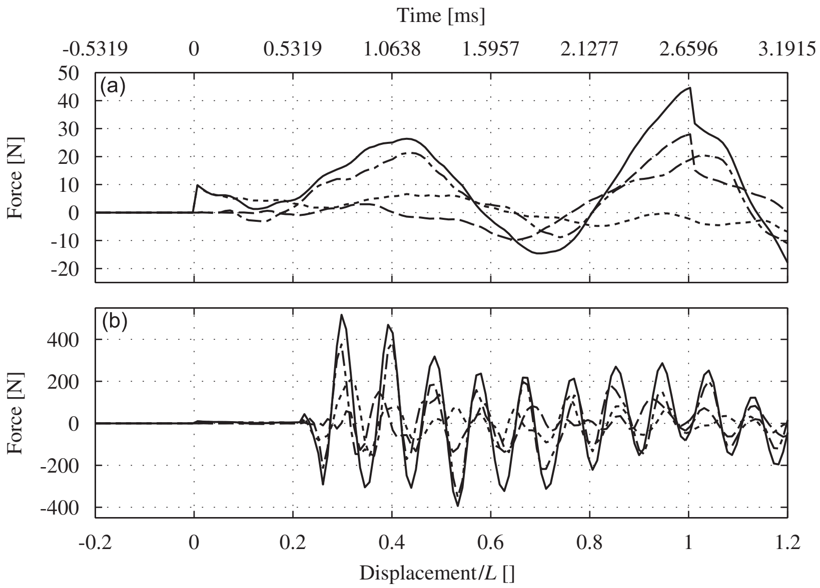

Fig. 9. Numerical force at the three supports when the oscillator travels with a velocity $v = v_{\mathrm{cr.}} = 188\,\mathrm{ms}^{-1}$ over a plane beam (a) or over a beam with a small bump (b): ··· first support, – · – · – second support, - - - third support and — sum of the three forces.

Finally, Fig. 9(a) shows the forces transmitted to the three supports when the oscillator moves across the beam at the critical velocity $v = v_{\mathrm{cr.}} = 188\,\mathrm{ms}^{-1}$. In this case, even though at the beginning the forces behave as with the slower speeds, the forces continue to be excited by the oscillator and the forces increase to 40 N. At the end of the beam is possible to see the unloading of the oscillator from the beam. Fig. 9(b) shows the results when the oscillator moves over the bump. The force impulse due to the road bump is now very large and the total force at the support reach 500 N.

5. Conclusions

A general method of dealing with an elastic system interacting with a moving dynamic load has been described. A new more general approach than has previously employed has been proposed to the force transmissibility and the relationship between force and velocity transmissibility has been investigated. Both the steady and moving system have been described in the state-space domain. The moving contact points have been accounted for by connecting the two sub-systems within a time-varying feedback formulation.

The methodology has been tested on the simple case of an oscillator travelling over a three-hinged beam. The methodology is simple to apply even with common software and permits the rapid calculation of the force transmitted to the ground under different working conditions.

One potential advantage of this approach is the possibility of using this model to control the force transmissibility using state-space feedback theory. The numerical simulation also permits the consideration of a moving body with time-varying speed and any road profile.

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